6.189 IAP 2007

Lecture 15

Cilk

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Design and Analysis of Dynamic Multithreaded Algorithms

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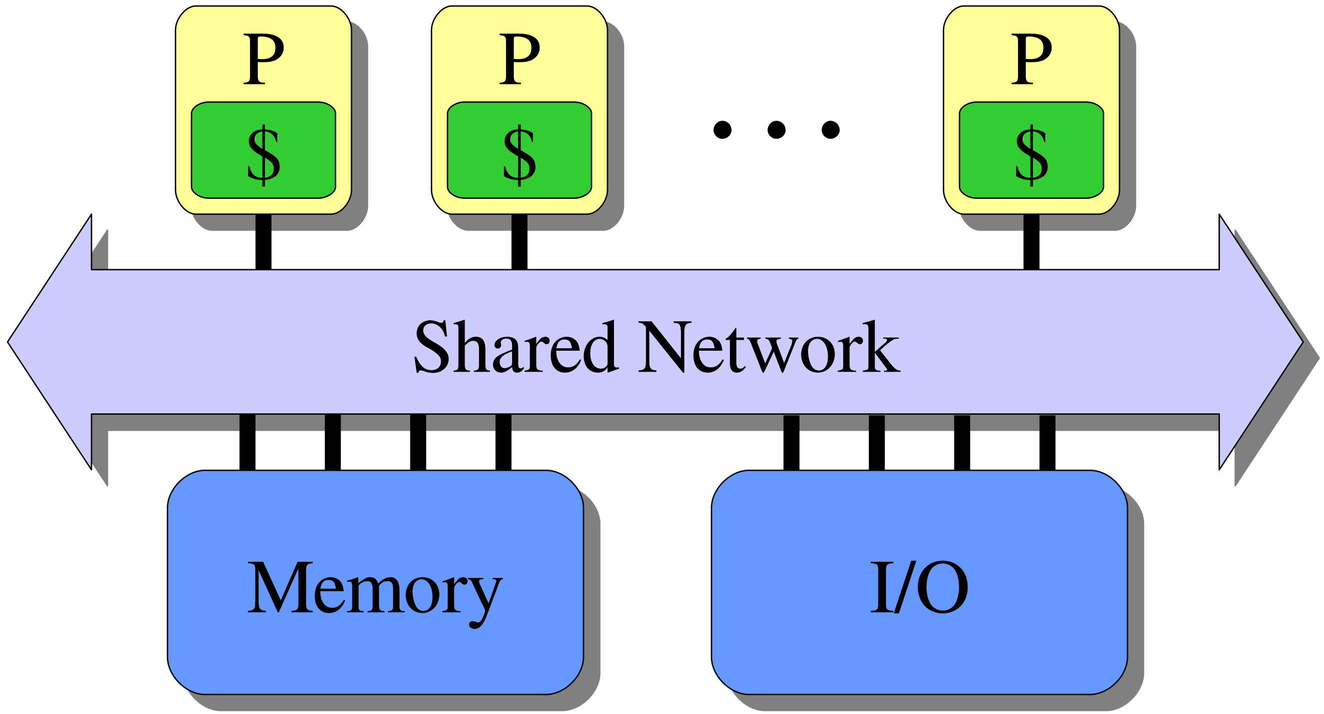
Shared-Memory Multiprocessor



- ***Symmetric multiprocessor (SMP)***
- ***Cache-coherent nonuniform memory architecture (CC-NUMA)***

Cilk

A C language for dynamic multithreading with a provably good runtime system.

Platforms

- Sun UltraSPARC Enterprise
- SGI Origin 2000
- Compaq/Digital Alphaserver
- Intel Pentium SMP's

Applications

- virus shell assembly
- graphics rendering
- *n*-body simulation
- ★Socrates and Cilkchess

Cilk automatically manages low-level aspects of parallel execution, including protocols, load balancing, and scheduling.

Fibonacci

```
int fib (int n) {
if (n<2) return (n);
  else {
    int x,y;
    x = fib(n-1);
    y = fib(n-2);
    return (x+y);
  }
}
```

C elision

Cilk code

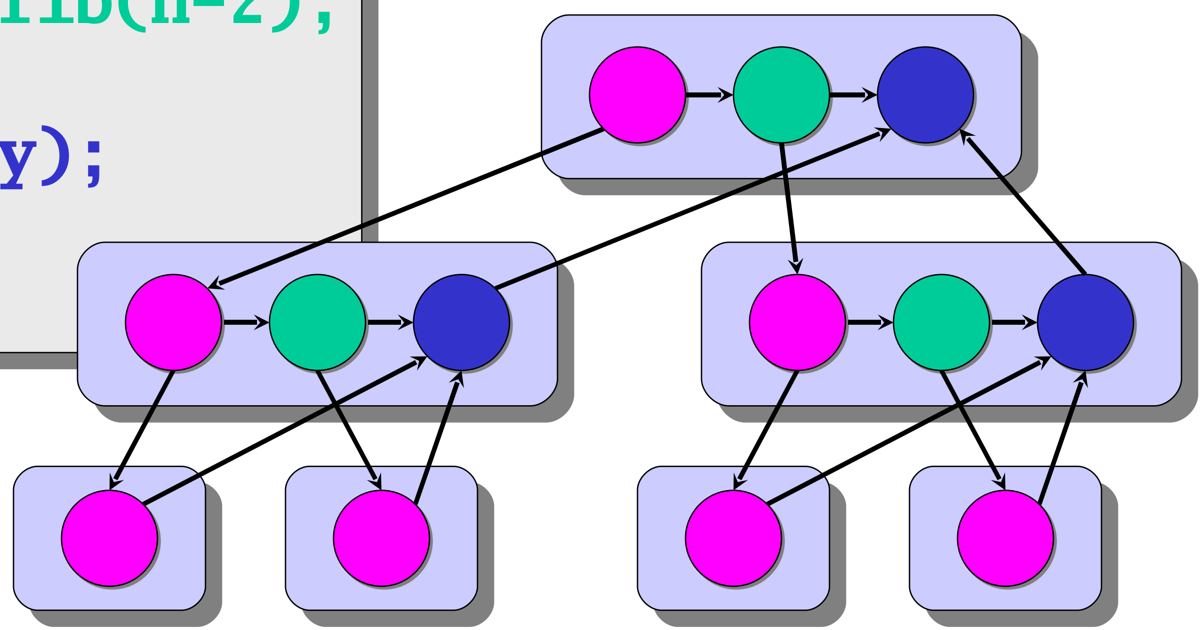
```
cilk int fib (int n) {
  if (n<2) return (n);
  else {
    int x,y;
    x = spawn fib(n-1);
    y = spawn fib(n-2);
    sync;
    return (x+y);
  }
}
```

Cilk is a ***faithful*** extension of C. A Cilk program's ***serial elision*** is always a legal implementation of Cilk semantics. Cilk provides ***no*** new data types.

Dynamic Multithreading

```
cilk int fib (int n) {
  if (n<2) return (n);
  else {
    int x,y;
    x = spawn fib(n-1);
    y = spawn fib(n-2);
    sync;
    return (x+y);
  }
}
```

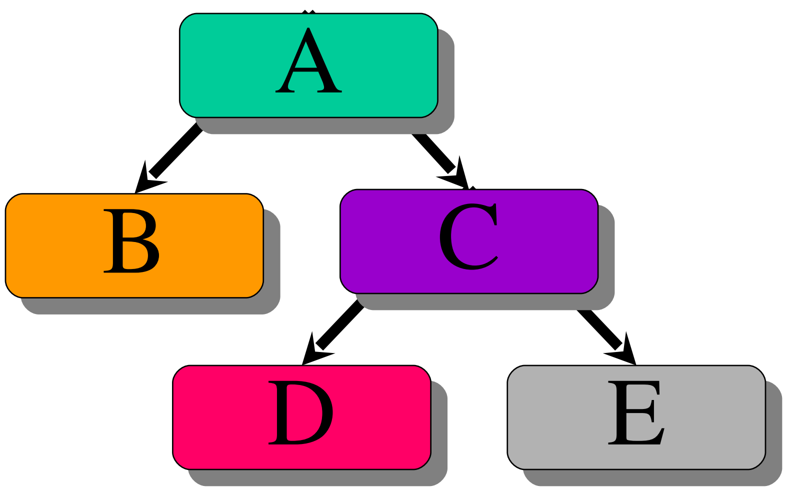
*The **computation dag** unfolds dynamically.*

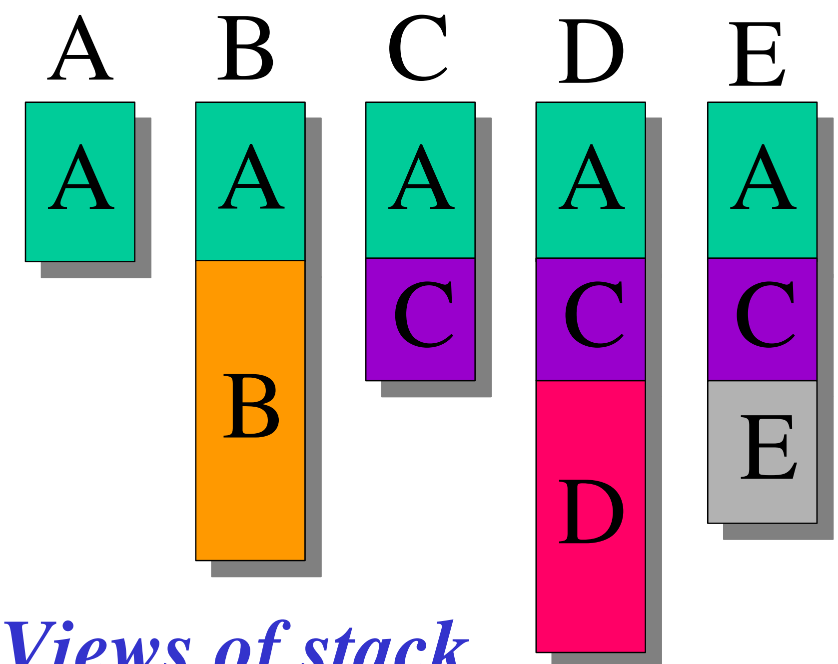


"Processor oblivious."

Cactus Stack

Cilk supports C's rule for pointers: A pointer to stack space can be passed from parent to child, but not from child to parent. (Cilk also supports **`malloc`**.)





Views of stack

Cilk's *cactus stack* supports several views in parallel.

Advanced Features

- Returned values can be incorporated into the parent frame using a delayed internal function called an ***inlet***:

```
int y;
inlet void foo (int x) {
  if (x > y) y = x;
}
…
spawn foo(bar(z));
```

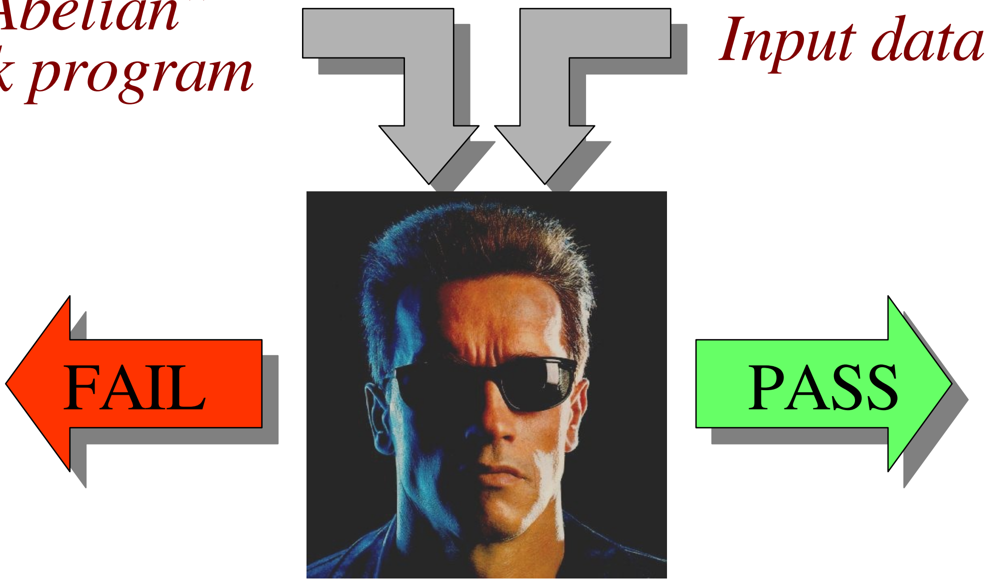
- Within an inlet, the **abort** keyword causes all other children of the parent frame to be terminated.
- The **SYNCHED** pseudovariable tests whether a **sync** would succeed.
- A Cilk library provides ***mutex locks*** for atomicity.

Debugging Support

The ***Nondeterminator*** debugging tool detects and localizes data-race bugs.

"Abelian" Cilk program

Input data set



Information localizing a data race.

Every scheduling produces the same result.

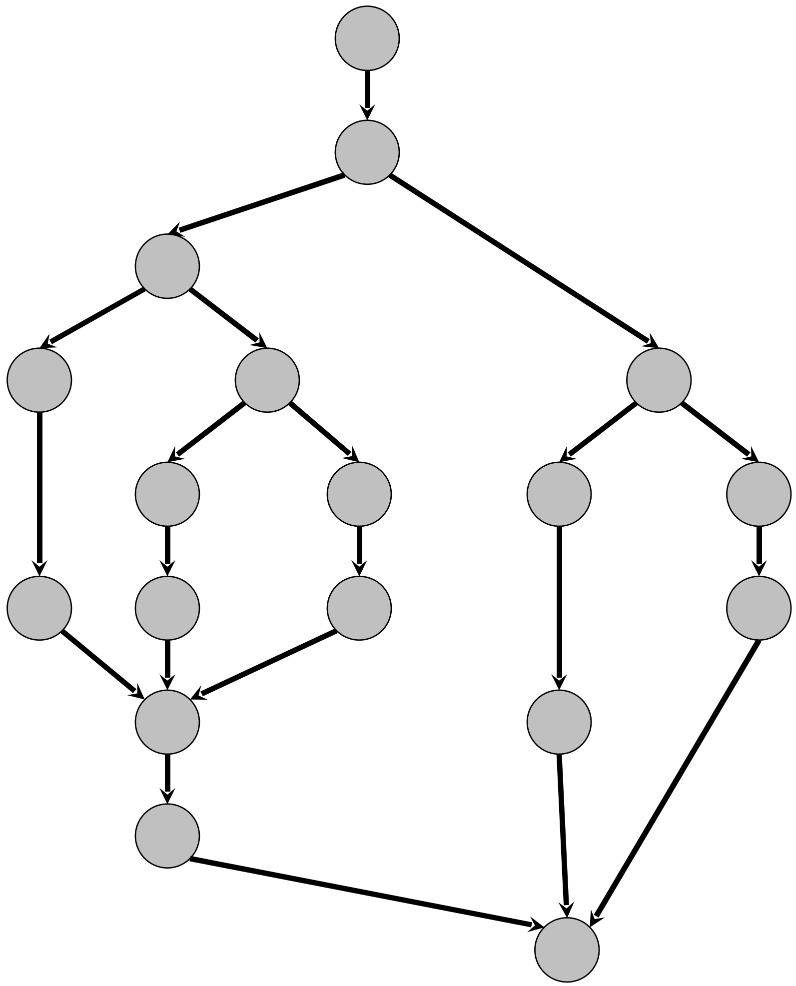
A ***data race*** occurs whenever a thread modifies a location and another thread, holding no locks in common, accesses the location simultaneously.

Outline

- Theory and Practice
- A Chess Lesson
- Fun with Algorithms
- Work Stealing
- Opinion & Conclusion

Algorithmic Complexity Measures

T_P = execution time on P processors



Algorithmic Complexity Measures

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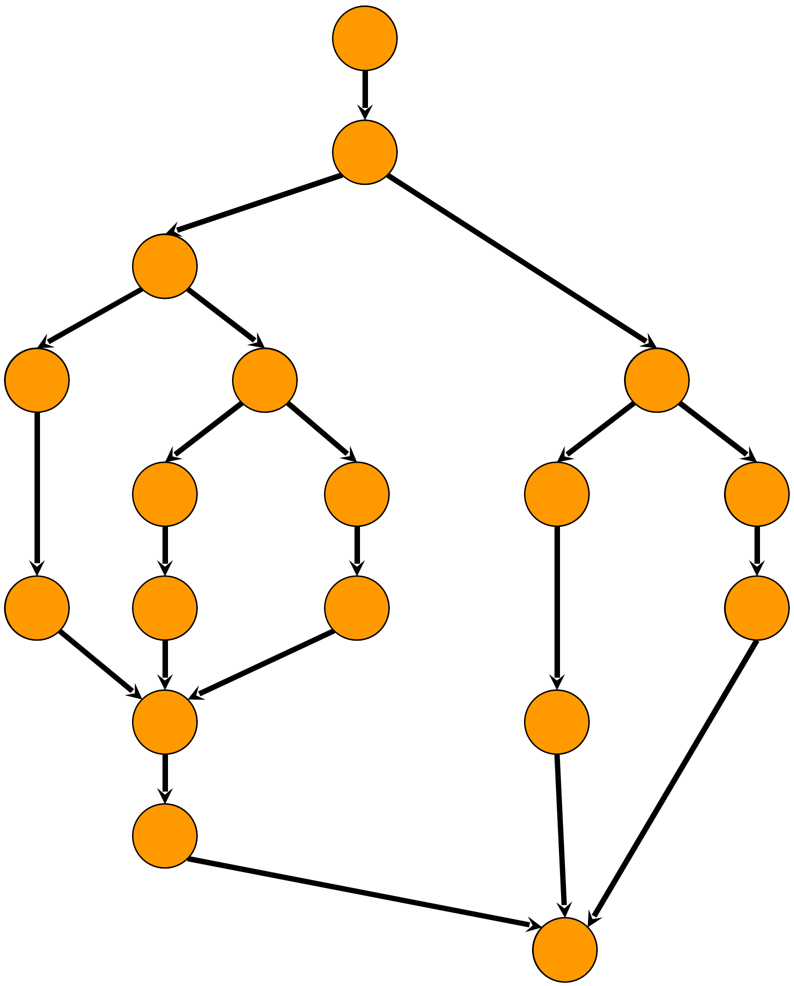
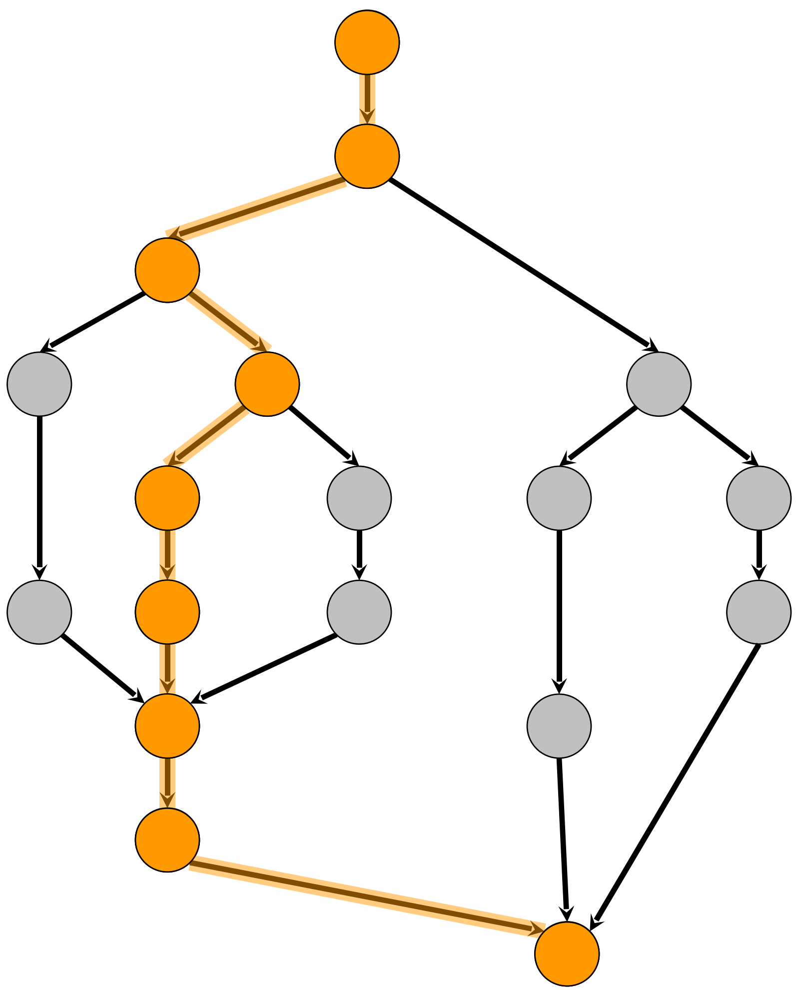


T_1 = ***work***

Algorithmic Complexity Measures

T_P = execution time on P processors



T_1 = ***work***

T_∞ = ***critical path***

Algorithmic Complexity Measures

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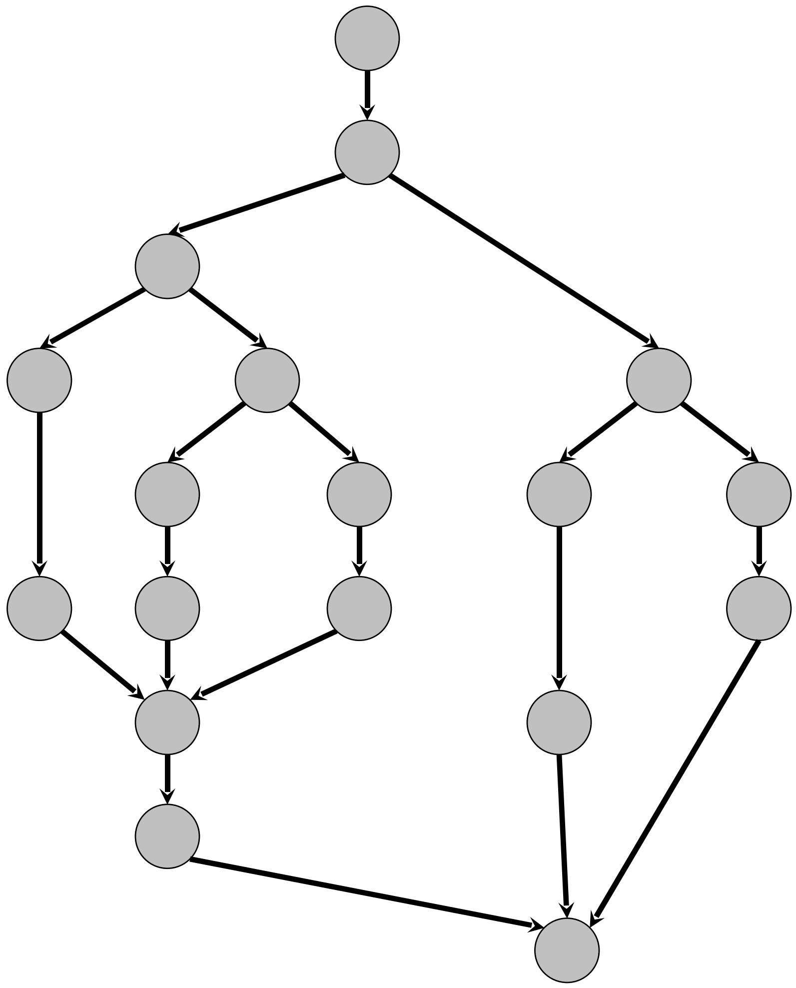

T_1 = ***work***

T_∞ = ***critical path***

Lower Bounds

- $T_P \geq T_1/P$
- $T_P \geq T_\infty$

Algorithmic Complexity Measures

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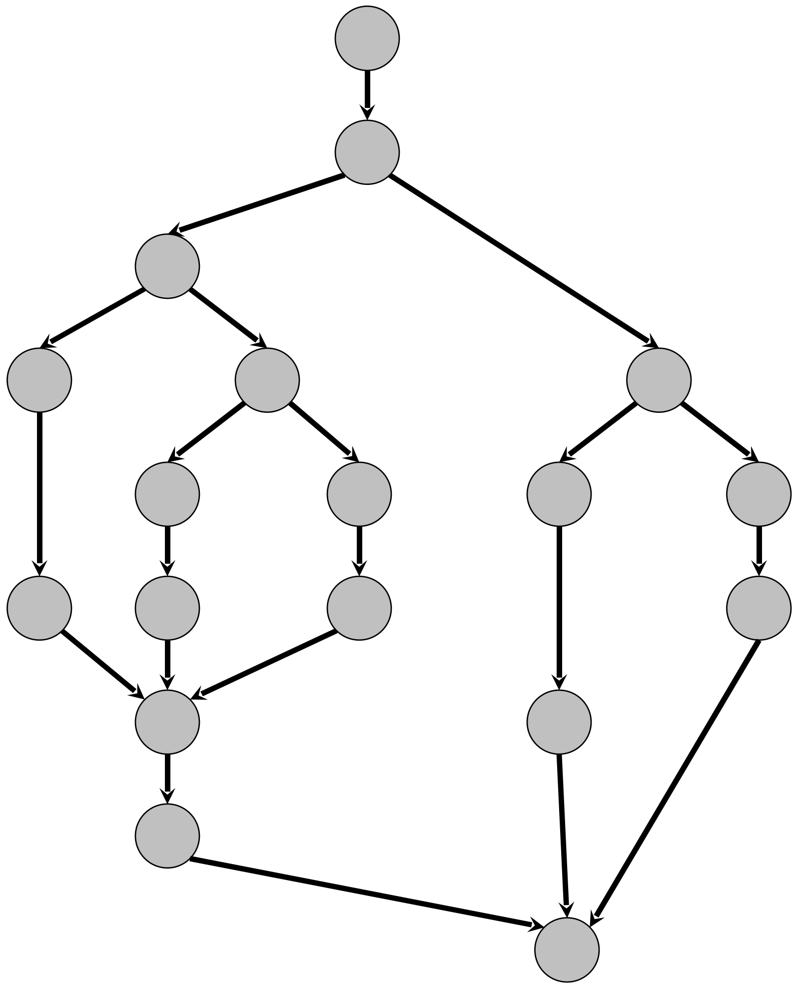


T_1 = ***work***

T_∞ = ***critical path***

Lower Bounds

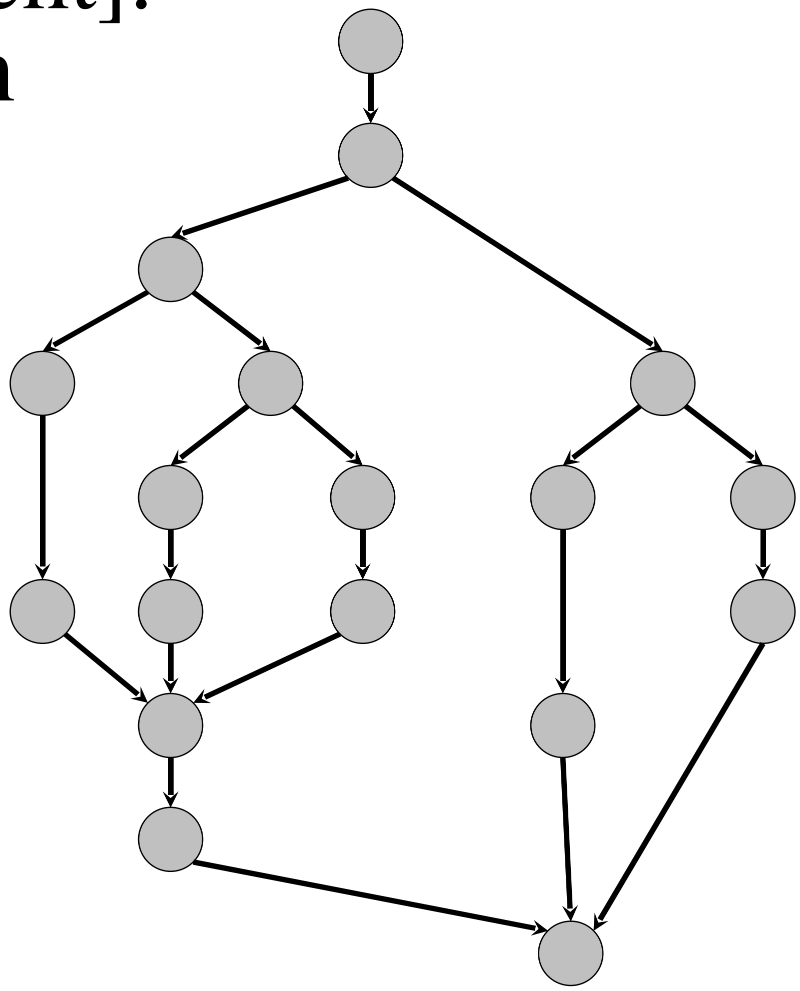
- $T_P \geq T_1/P$
- $T_P \geq T_\infty$

T_1/T_P = ***speedup***

T_1/T_∞ = ***parallelism***

Greedy Scheduling

Theorem [Graham & Brent]:
There exists an execution with $T_P \le T_1/P + T_\infty$.

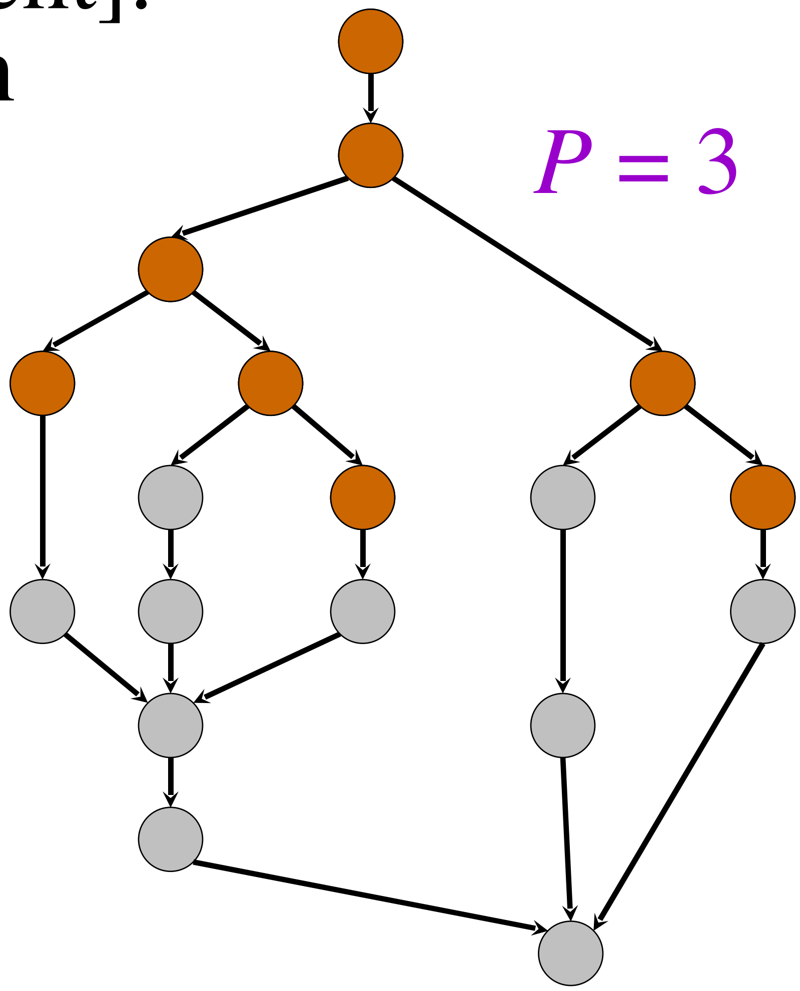


Greedy Scheduling

Theorem [Graham & Brent]:
There exists an execution with $T_P \leq T_1/P + T_\infty$.

Proof. At each time step, …

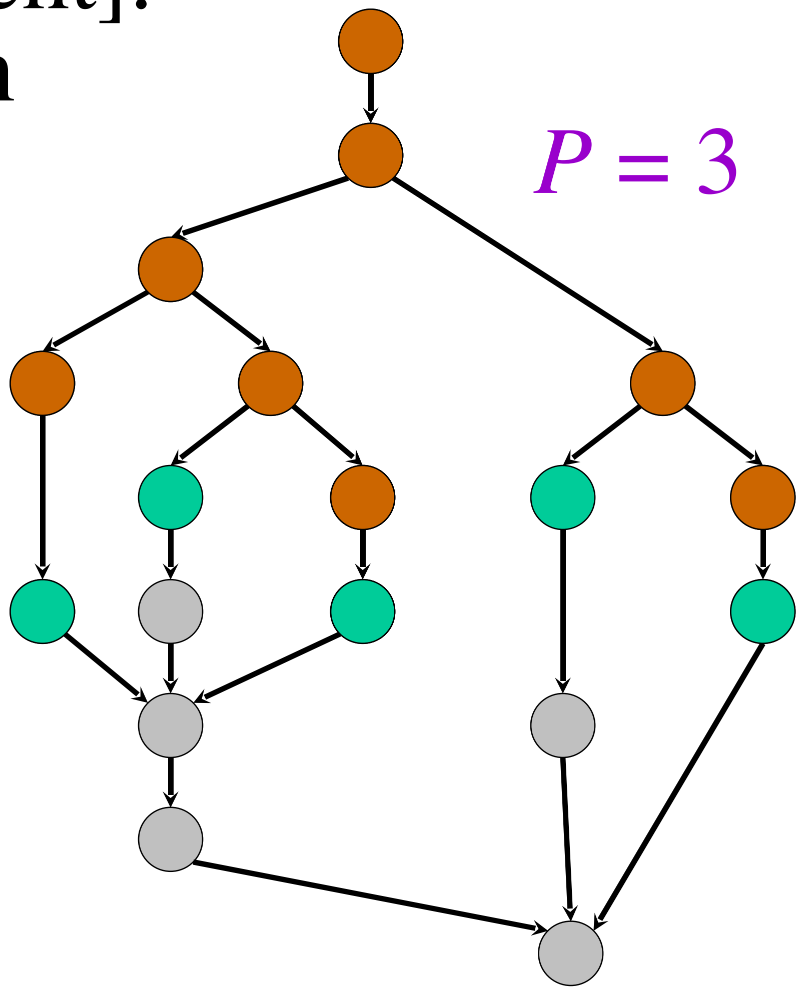
$P = 3$



Greedy Scheduling

Theorem [Graham & Brent]: There exists an execution with $T_P \le T_1/P + T_\infty$.

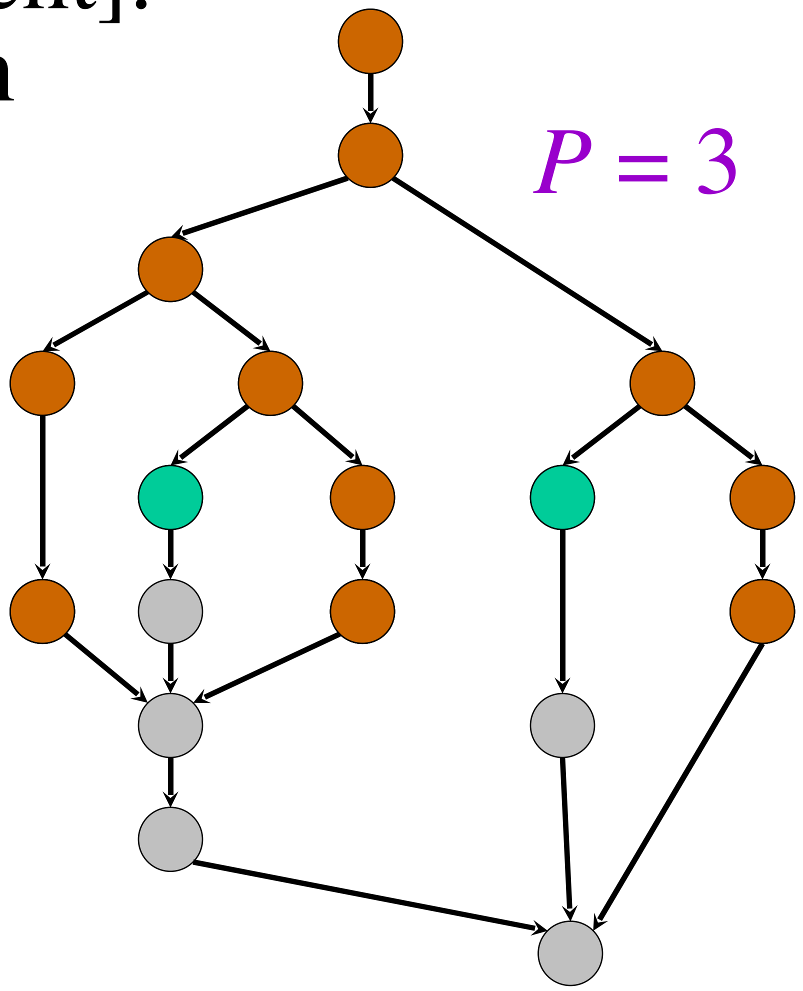
Proof. At each time step, if at least P tasks are ready, …



Greedy Scheduling

Theorem [Graham & Brent]: There exists an execution with $T_P \leq T_1/P + T_\infty$.

Proof. At each time step, if at least P tasks are ready, execute P of them.



Greedy Scheduling

Theorem [Graham & Brent]: There exists an execution with $T_P \leq T_1/P + T_\infty$.

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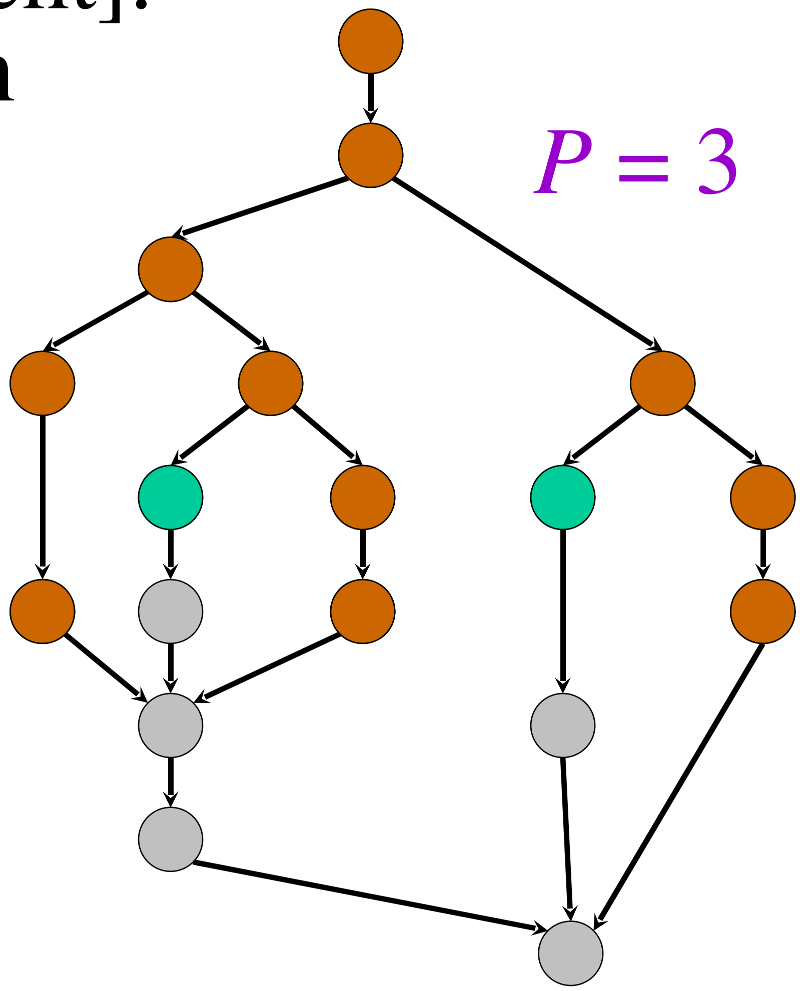

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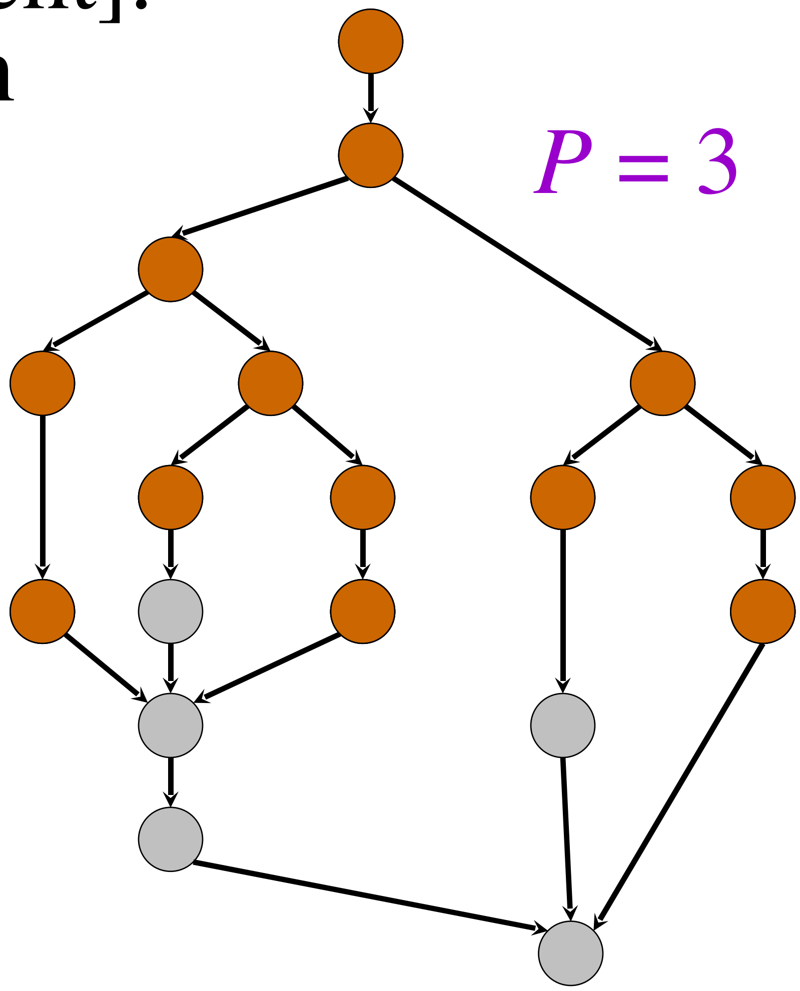

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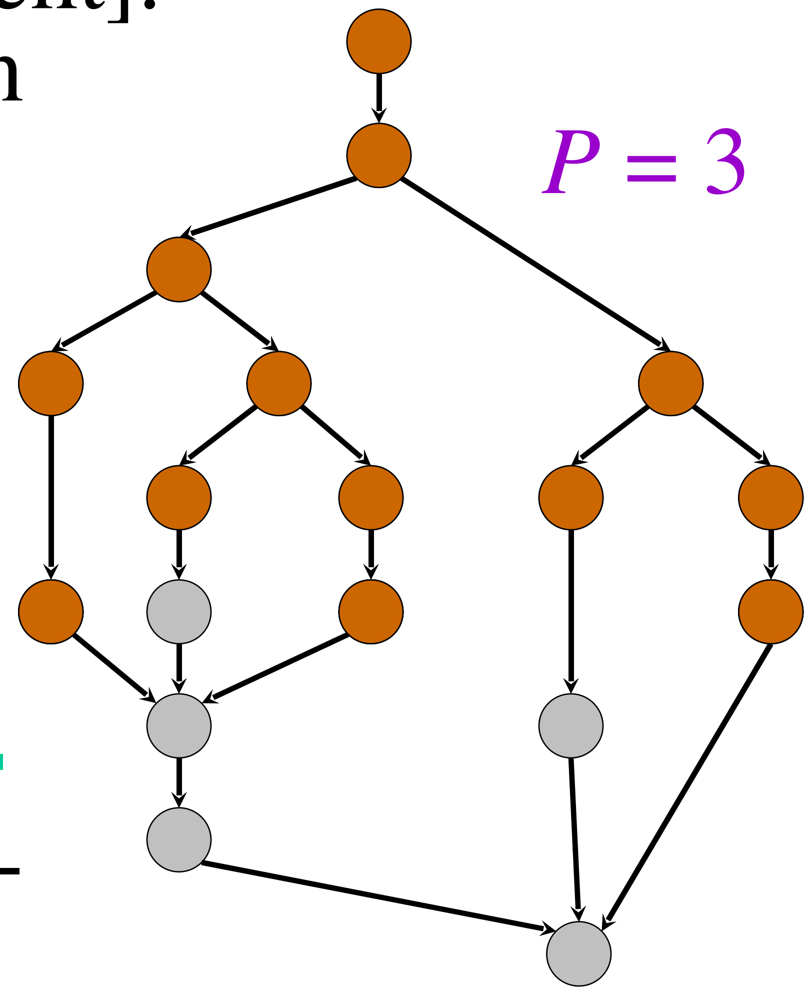

Corollary: Linear speed-up when $P \le T_1/T_\infty$.

Cilk Performance

Cilk's "***work-stealing***" scheduler achieves

- $T_P = T_1/P + O(T_\infty)$ expected time (provably);
- $T_P \approx T_1/P + T_\infty$ time (empirically).

Near-perfect linear speedup if $P \leq T_1/T_\infty$.

Instrumentation in Cilk provides accurate measures of T_1 and T_∞ to the user.

The average cost of a **spawn** in Cilk-5 is only 2–6 times the cost of an ordinary C function call, depending on the platform.

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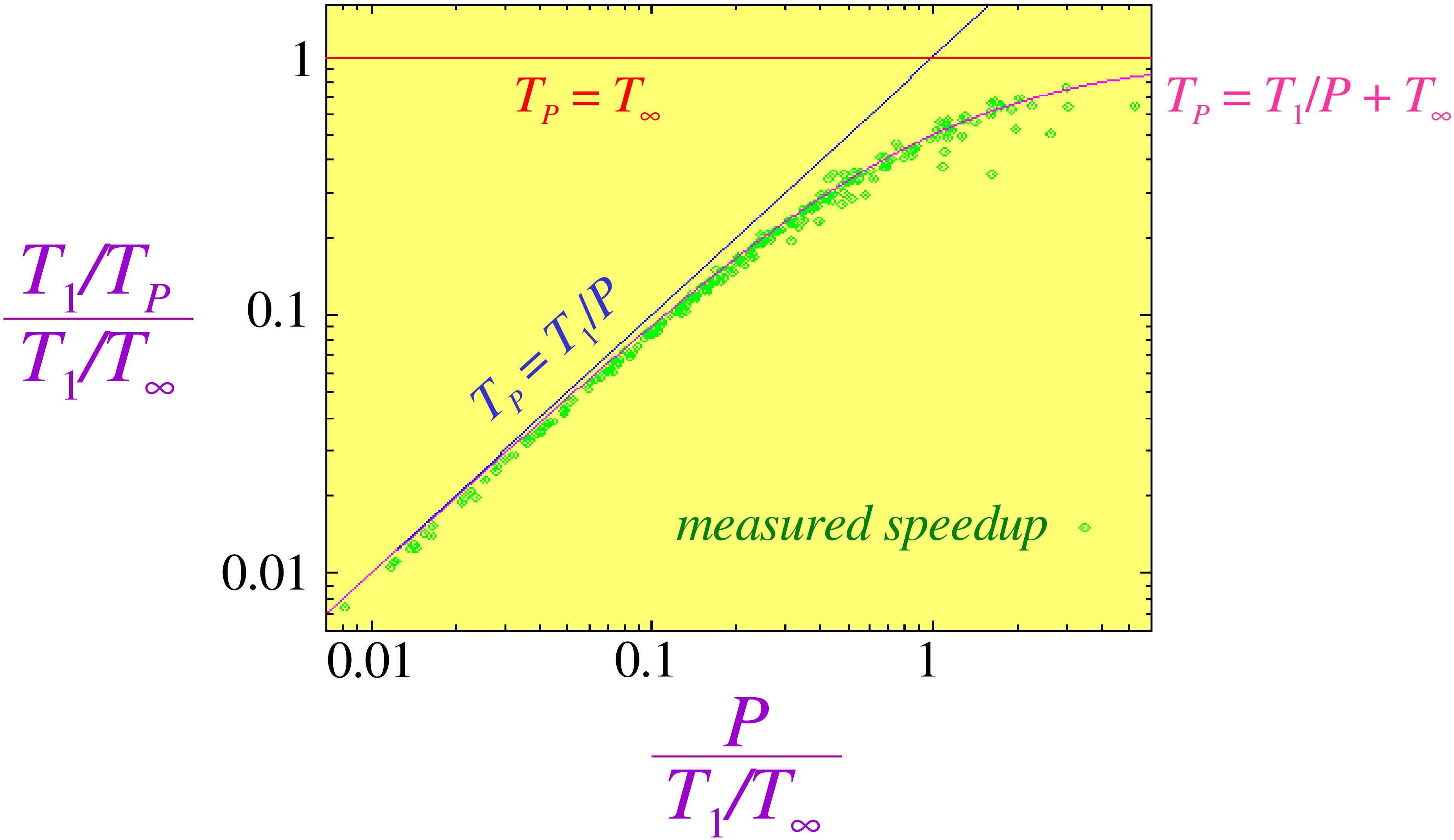
Cilk Chess Programs

- ***Socrates*** placed 3rd in the 1994 International Computer Chess Championship running on NCSA's 512-node Connection Machine CM5.
- ***Socrates 2.0*** took 2nd place in the 1995 World Computer Chess Championship running on Sandia National Labs' 1824-node Intel Paragon.
- ***Cilkchess*** placed 1st in the 1996 Dutch Open running on a 12-processor Sun Enterprise 5000. It placed 2nd in 1997 and 1998 running on Boston University's 64-processor SGI Origin 2000.
- ***Cilkchess*** tied for 3rd in the 1999 WCCC running on NASA's 256-node SGI Origin 2000.

Socrates Normalized Speedup



Socrates Speedup Paradox

Original program

$T_{32} = 65$ seconds

Proposed program

$T'_{32} = 40$ seconds

$$T_P \approx T_1/P + T_\infty$$

$T_1 = 2048$ seconds
$T_\infty = 1$ second

$T'_1 = 1024$ seconds
$T'_\infty = 8$ seconds

$T_{32} = 2048/32 + 1$
$= 65$ seconds

$T'_{32} = 1024/32 + 8$
$= 40$ seconds

$T_{512} = 2048/512 + 1$
$= 5$ seconds

$T'_{512} = 1024/512 + 8$
$= 10$ seconds

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Matrix Multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

C A B

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Recursive Matrix Multiplication

Divide and conquer on $n \times n$ matrices.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{bmatrix} + \begin{bmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix}$$

8 multiplications of $(n/2) \times (n/2)$ matrices.
1 addition of $n \times n$ matrices.

Matrix Multiplication in Cilk

```
cilk Mult(*C,*A,*B,n)
{ float T[n][n];
  h base case & partition matrices i
  spawn Mult(C11,A11,B11,n/2);
  spawn Mult(C12,A11,B12,n/2);
  spawn Mult(C22,A21,B12,n/2);
  spawn Mult(C21,A21,B11,n/2);
  spawn Mult(T11,A12,B21,n/2);
  spawn Mult(T12,A12,B22,n/2);
  spawn Mult(T22,A22,B22,n/2);
  spawn Mult(T21,A22,B21,n/2);
  sync;
  spawn Add(C,T,n);
  sync;
  return;
}
```

$C = AB$

(Coarsen base cases for efficiency.)

$C = C + T$

```
cilk Add(*C,*T,n)
{    h base case & partition matrices i
     spawn Add(C11,T11,n/2);
     spawn Add(C12,T12,n/2);
     spawn Add(C21,T21,n/2);
     spawn Add(C22,T22,n/2);
     sync;
     return;
}
```

Analysis of Matrix Addition

```
cilk Add(*C,*T,n)
{    h base case & partition matrices i
     spawn Add(C11,T11,n/2);
     spawn Add(C12,T12,n/2);
     spawn Add(C21,T21,n/2);
     spawn Add(C22,T22,n/2);
     sync;
     return;
}
```

Work: $A_1(n) = 4\,A_1(n/2) + $ (1)

(n^2)

Critical path: $A_\infty(n) \equiv A_\infty(n/2) + $ (1)

$=$ (lg n)

Analysis of Matrix Multiplication

Work:

$$M_1(n) = 8\,M_1(n/2) + \quad (n^2)$$
$$= \quad (n^3)$$

Critical path:

$$M_\infty(n) = M_\infty(n/2) + \quad (\lg n)$$
$$= \quad (\lg^2 n)$$

Parallelism:

$$\frac{M_1(n)}{M_\infty(n)} = \quad (n^3/\lg^2 n)$$

For 1000 £ 1000 matrices, parallelism ¼ 10^7.

Stack Temporaries

```
cilk Mult(*C,*A,*B,n)
{ float T[n][n];
   h base case & partition matrices i
   spawn Mult(C11,A11,B11,n/2);
   spawn Mult(C12,A11,B12,n/2);
   spawn Mult(C22,A21,B12,n/2);
   spawn Mult(C21,A21,B11,n/2);
   spawn Mult(T11,A12,B21,n/2);
   spawn Mult(T12,A12,B22,n/2);
   spawn Mult(T22,A22,B22,n/2);
   spawn Mult(T21,A22,B21,n/2);
   sync;
   spawn Add(C,T,n);
   sync;
   return;
}
```

In modern hierarchical-memory microprocessors, memory accesses are so expensive that minimizing storage often yields higher performance.

No-Temp Matrix Multiplication

```
cilk Mult2(*C,*A,*B,n)
{ // C = C + A * B
   ⟨base case & partition matrices⟩
   spawn Mult2(C11,A11,B11,n/2);
   spawn Mult2(C12,A11,B12,n/2);
   spawn Mult2(C22,A21,B12,n/2);
   spawn Mult2(C21,A21,B11,n/2);
   sync;
   spawn Mult2(C21,A22,B21,n/2);
   spawn Mult2(C22,A22,B22,n/2);
   spawn Mult2(C12,A12,B22,n/2);
   spawn Mult2(C11,A12,B21,n/2);
   sync;
   return;
}
```

Saves space at the expense of critical path.

Analysis of No-Temp Multiply

Work: $M_1(n) = \quad (n^3)$

Critical path: $M_\infty(n) = 2\,M_\infty(n/2) + \quad (1)$

$= \quad (n)$

Parallelism: $\dfrac{M_1(n)}{M_\infty(n)} = \quad (n^2)$

For 1000 £ 1000 matrices, parallelism ¼ 10^6.
Faster in practice.

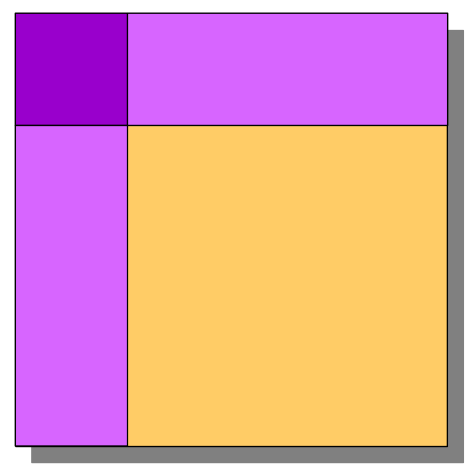
Ordinary Matrix Multiplication

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

IDEA: Spawn n^2 inner products in parallel. Compute each inner product in parallel.

Work: (n^3)
Critical path: $(\lg n)$
Parallelism: $(n^3/\lg n)$

BUT, this algorithm exhibits poor locality and does not exploit the cache hierarchy of modern microprocessors.



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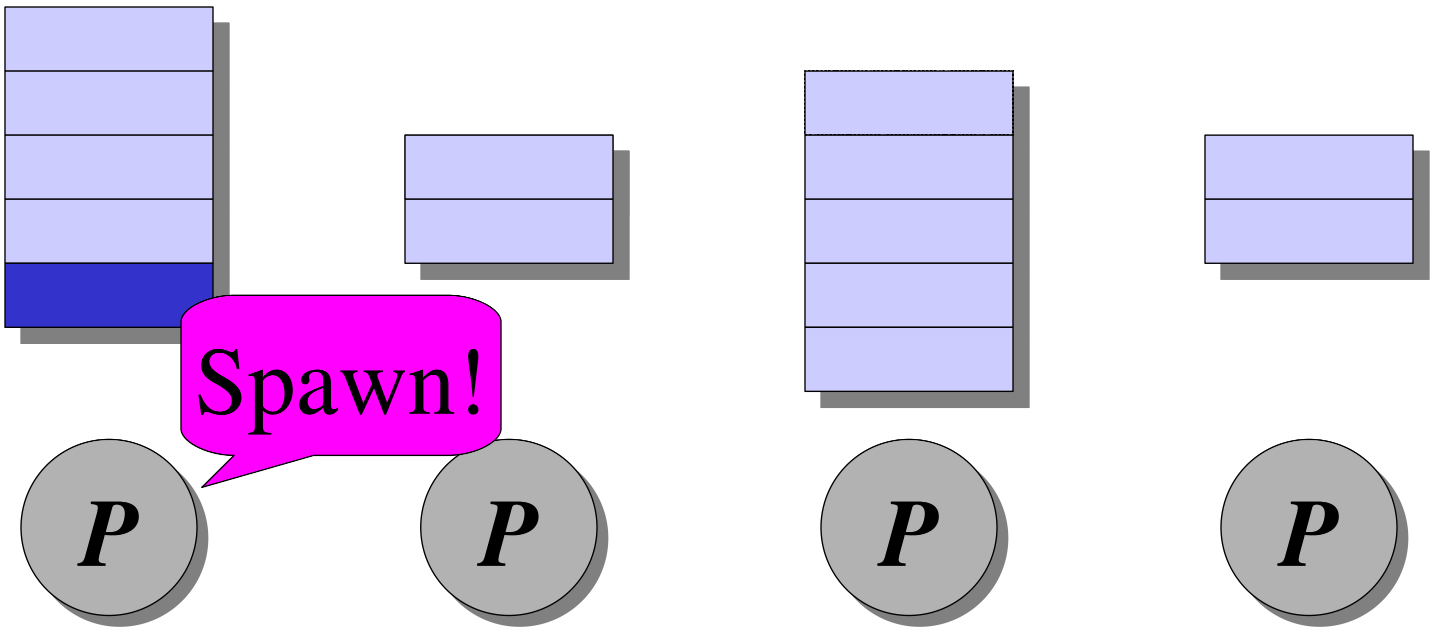
Cilk's Work-Stealing Scheduler

Each processor maintains a ***work deque*** of ready threads, and it manipulates the bottom of the deque like a stack.



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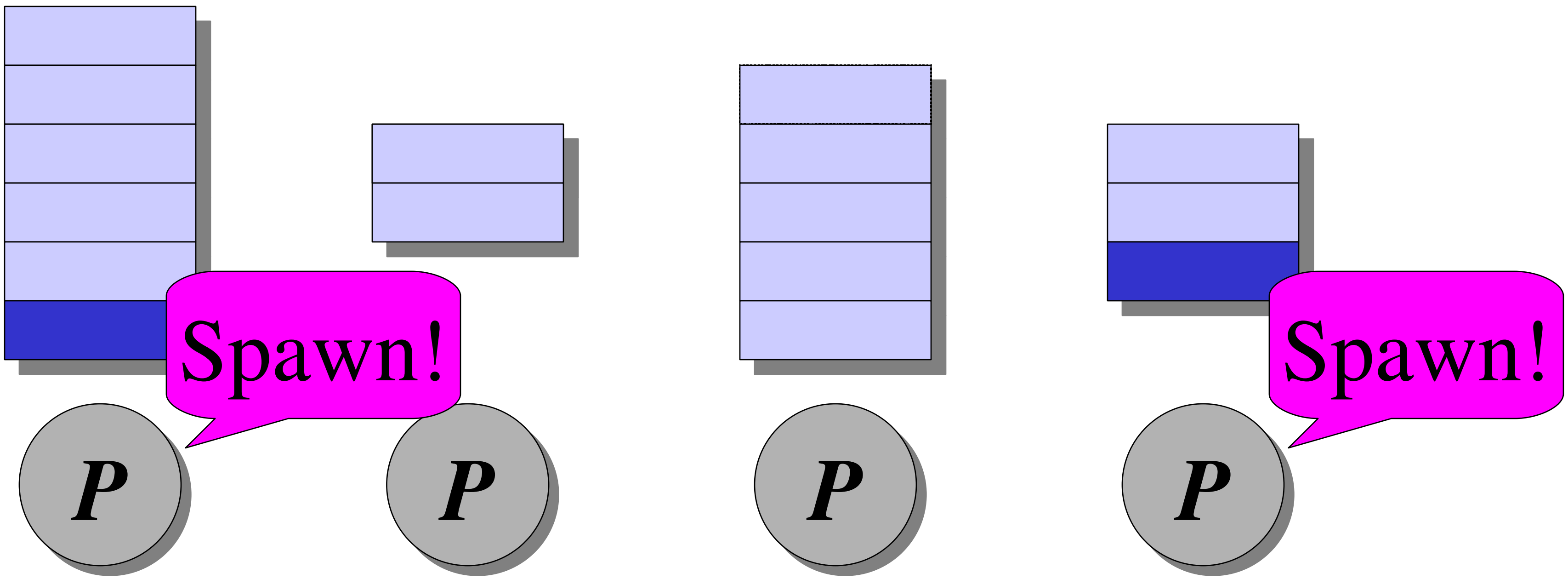

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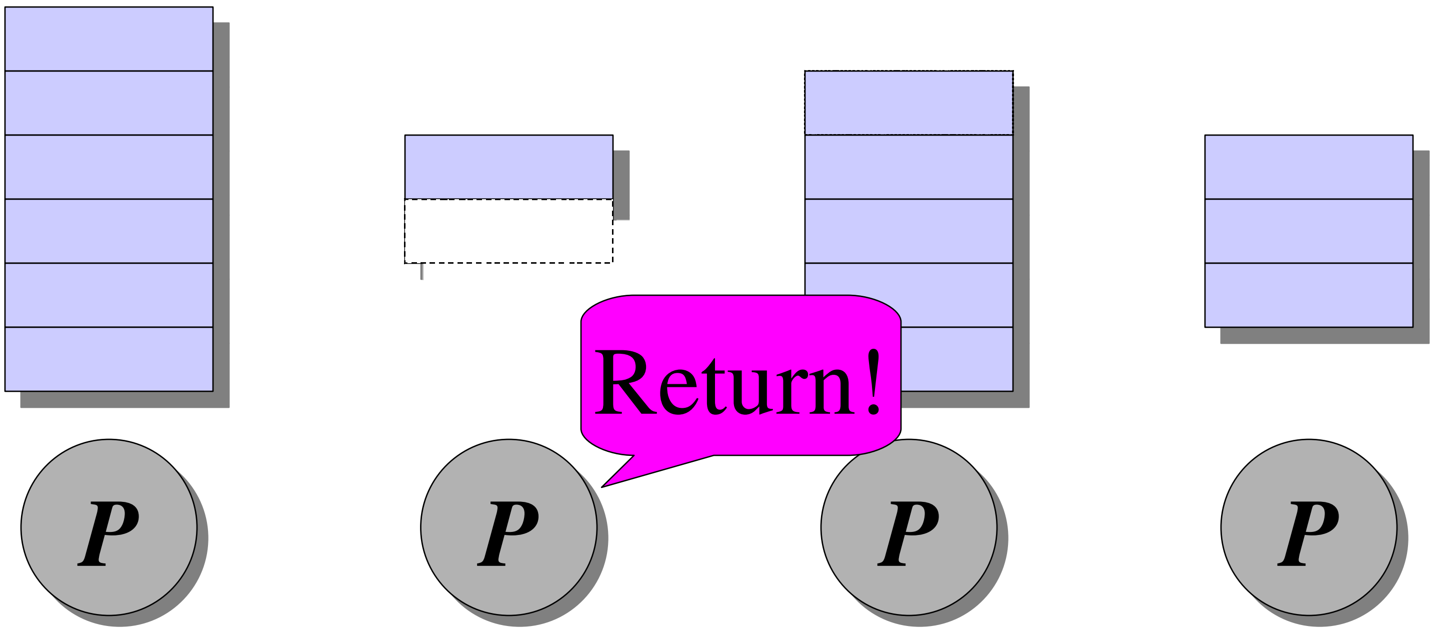

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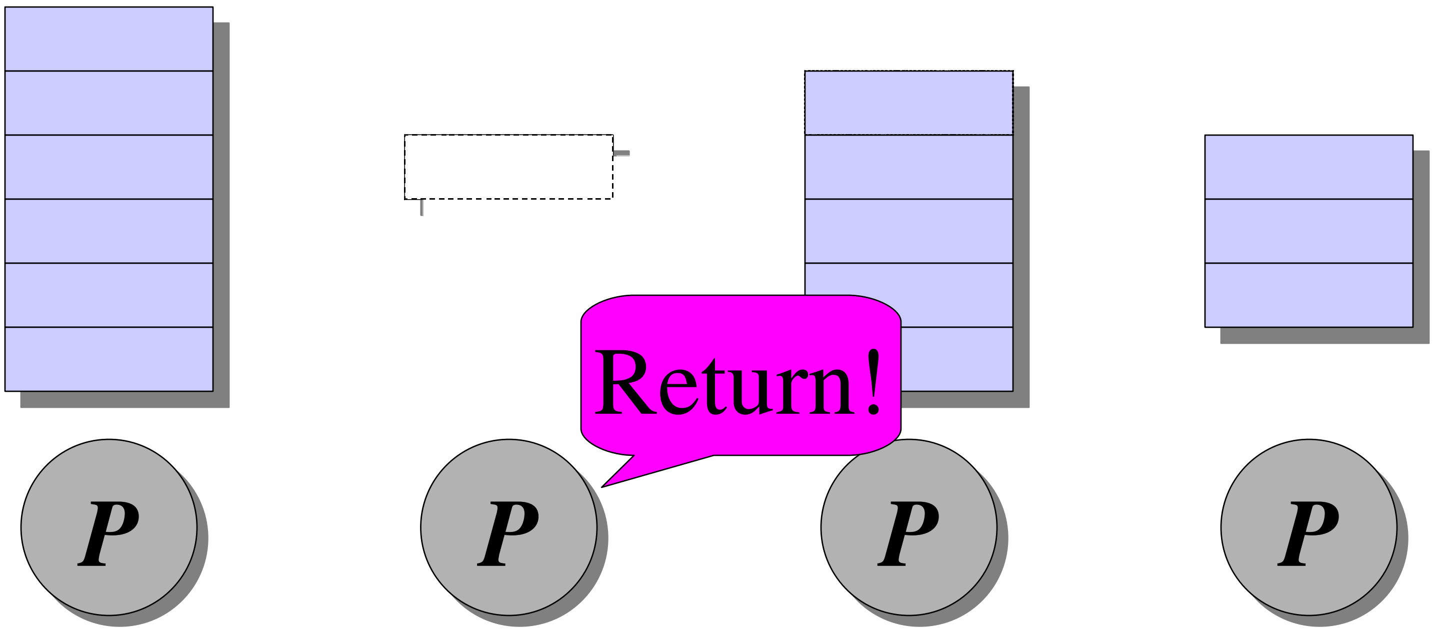

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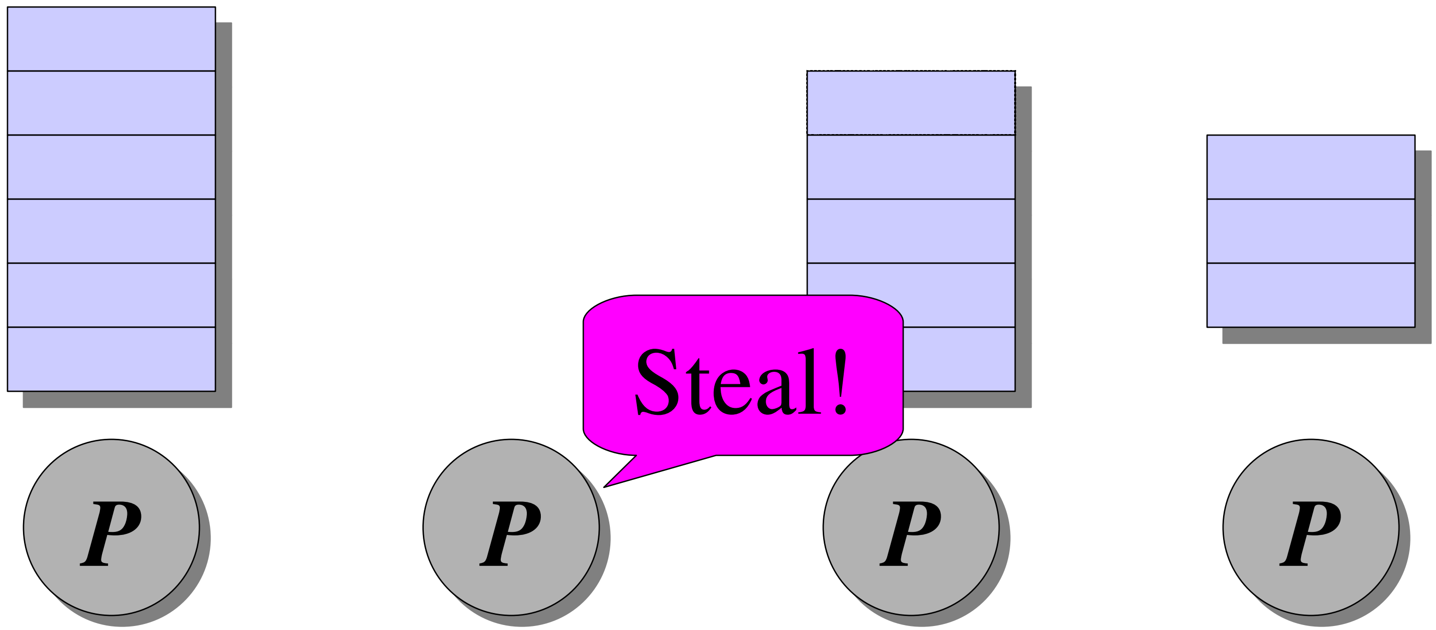
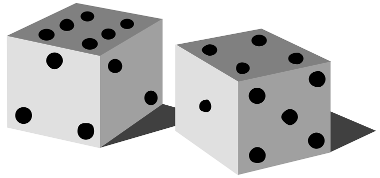


When a processor runs out of work, it ***steals*** a thread from the top of a ***random*** victim's deque.



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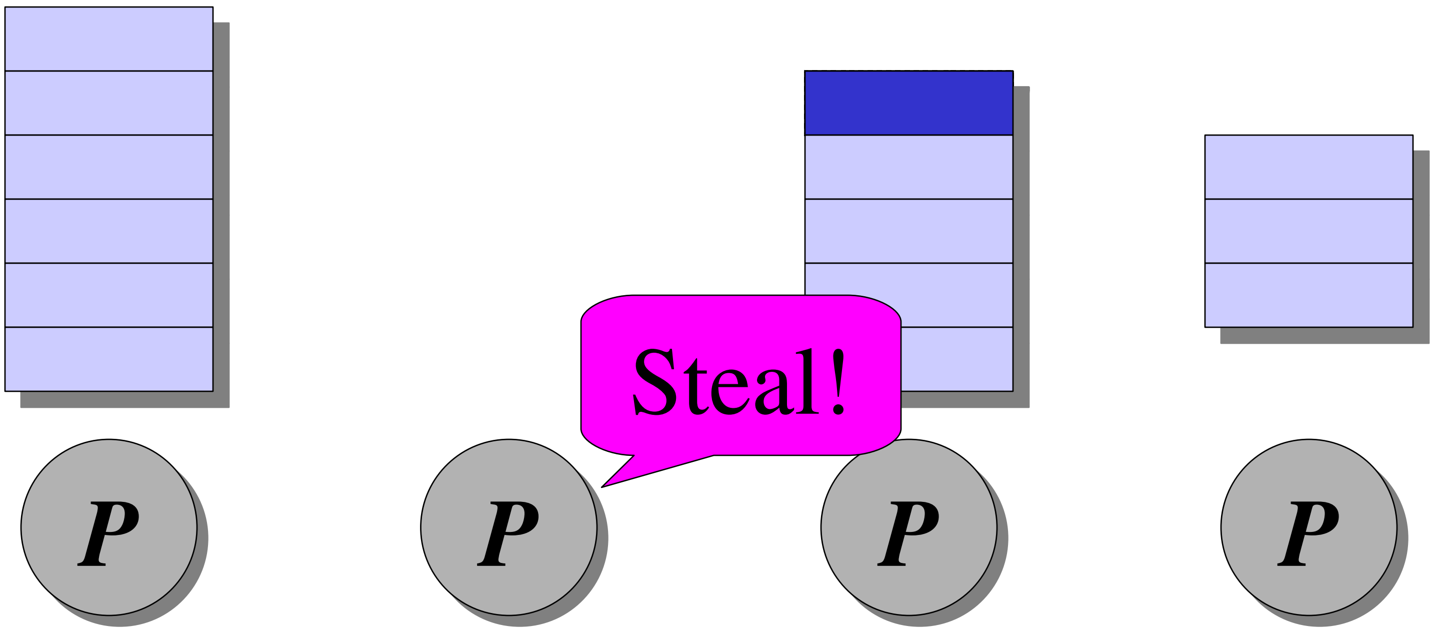

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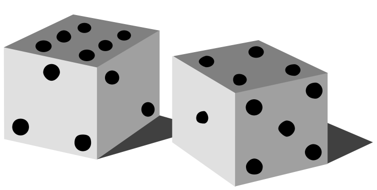

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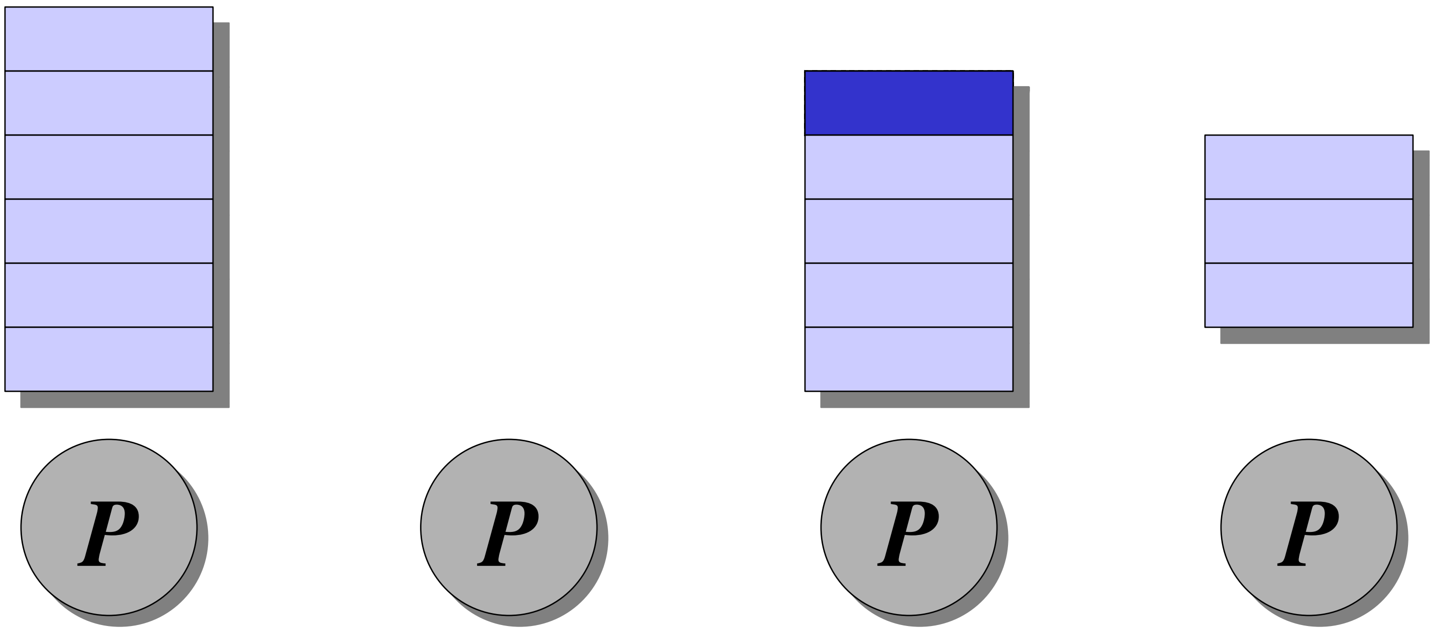

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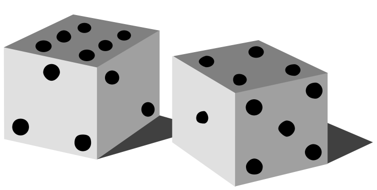

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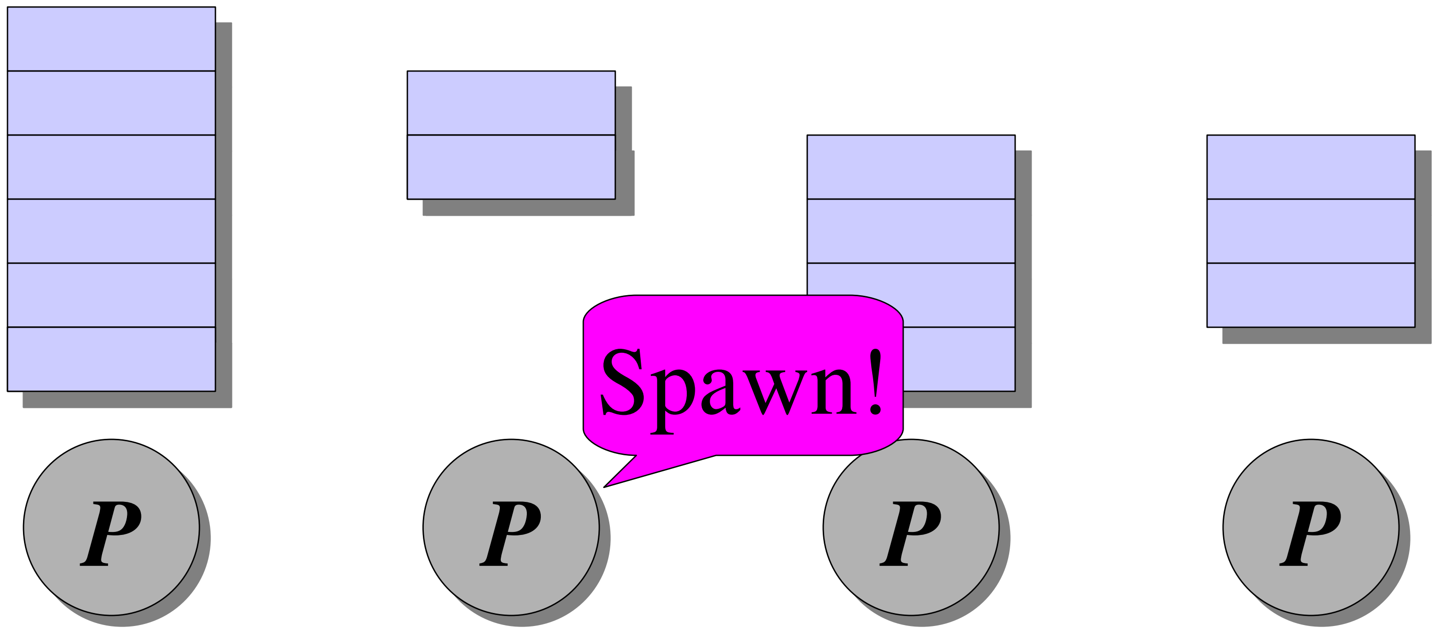

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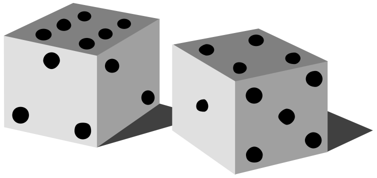

Performance of Work-Stealing

Theorem: A work-stealing scheduler achieves an expected running time of

$$T_P \leq T_1/P + O(T_1)$$

on P processors.

Pseudoproof. A processor is either ***working*** or ***stealing***. The total time all processors spend working is T_1. Each steal has a $1/P$ chance of reducing the critical-path length by 1. Thus, the expected number of steals is $O(PT_1)$. Since there are P processors, the expected time is $(T_1 + O(PT_1))/P = T_1/P + O(T_1)$.

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Data Parallelism

- ☺ High level
- ☺ Intuitive
- ☺ Scales up

- ☹ Conversion costs
- ☹ Doesn't scale down
- ☹ Antithetical to caches
- ☹ Two-source problem
- ☹ Performance from tuned libraries

Example:

```
C = A + B;
D = A – B;
```

6 memory references, rather than 4.

Message Passing

- ☺ Scales up
- ☺ No compiler support needed
- ☺ Large inertia
- ☺ Runs anywhere

- ☹ Coarse grained
- ☹ Protocol intensive
- ☹ Difficult to debug
- ☹ Two-source problem
- ☹ Performance from tuned libraries

Shared memory ← *easier* In-core

↓ *harder* ↓ *harder*

Distributed memory ← *easier* Out-of-core

Conventional (Persistent) Multithreading

- ☺ Scales up and down
- ☺ No compiler support needed
- ☺ Large inertia
- ☺ Evolutionary

- ☹ Clumsy
- ☹ No load balancing
- ☹ Coarse-grained control
- ☹ Protocol intensive
- ☹ Difficult to debug

Parallelism for *programs*, not *procedures*.

Dynamic Multithreading

- ☺ High-level linguistic support for fine-grained control and data manipulation.
- ☺ Algorithmic programming model based on work and critical path.
- ☺ Easy conversion from existing codes.
- ☺ Applications that scale up and down.
- ☺ Processor-oblivious machine model that can be implemented in an adaptively parallel fashion.
- ☹ Doesn't support a "program model" of parallelism.

Current Research

- We are currently designing ***jCilk***, a Java-based language that fuses dynamic and persistent multithreading in a single linguistic framework.
- A key piece of algorithmic technology is an ***adaptive task scheduler*** that guarantees fair and efficient execution.
- ***Hardware transactional memory*** appears to simplify thread synchronization and improve performance compared with locking.
- The *Nondeterminator 3* will be the first parallel data-race detector to guarantee both efficiency and linear speed-up.

Cilk Contributors

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…plus many MIT students and SourceForgers.

World Wide Web

Cilk source code, programming examples, documentation, technical papers, tutorials, and up-to-date information can be found at:

http://supertech.csail.mit.edu/cilk

Download CILK Today!

Research Collaboration

Cilk is now being used at many universities for teaching and research:

MIT, Carnegie-Mellon, Yale, Texas, Dartmouth, Alabama, New Mexico, Tel Aviv, Singapore.

We need help in maintaining, porting, and enhancing Cilk's infrastructure, libraries, and application code base. If you are interested, send email to:

cilk-support@supertech.lcs.mit.edu

Warning: *We are not organized!*

Cilk-5 Benchmarks

Program	Size	T_1	T_∞	T_1/T_∞	T_1/T_S	T_8	T_1/T_8
`blockedmul`	1024	29.9	.0046	6783	1.05	4.29	7.0
`notempmul`	1024	29.7	.0156	1904	1.05	3.9	7.6
`strassen`	1024	20.2	.5662	36	1.01	3.54	5.7
`queens`	22	150.0	.0015	96898	0.99	18.8	8.0
`cilksort*`	4.1M	5.4	.0048	1125	1.21	0.9	6.0
`knapsack`	30	75.8	.0014	54143	1.03	9.5	8.0
`lu`	2048	155.8	.4161	374	1.02	20.3	7.7
`cholesky*`	1.02M	1427.0	3.4	420	1.25	208	6.9
`heat`	2M	62.3	.16	384	1.08	9.4	6.6
`fft`	1M	4.3	.002	2145	0.93	0.77	5.6
`barnes-hut`	65536	124.0	.15	853	1.02	16.5	7.5

All benchmarks were run on a Sun Enterprise 5000 SMP with 8 167-megahertz UltraSPARC processors. All times are in seconds, repeatable to within 10%.

Ease of Programming

	Original C	Cilk	SPLASH-2
lines	1861	2019	2959
lines	0	158	1098
diff *lines*	0	463	3741
T_1/T_8	1	7.5	7.2
T_1/T_S	1	1.024	1.099
T_S/T_8	1	7.3	6.6

Barnes-Hut application for 64K particles running on a 167-MHz Sun Enterprise 5000.

ICFP Programming Contest

- An 8-person Cilk team won **FIRST PRIZE** in the 1998 Programming Contest sponsored by the International Conference on Functional Programming.
- Our Cilk "***Pousse***" program was undefeated among the 49 entries. (Half the entries were coded in C.)
- Parallelizing our program to run on 4 processors took less than 1% of our effort, but it gave us more than a 3.5× performance advantage over our competitors.
- The ICFP Tournament Directors cited Cilk as ***"the superior programming tool of choice for discriminating hackers."***

- For details, see:

`http://supertech.lcs.mit.edu/~pousse`

Whither Functional Programming?

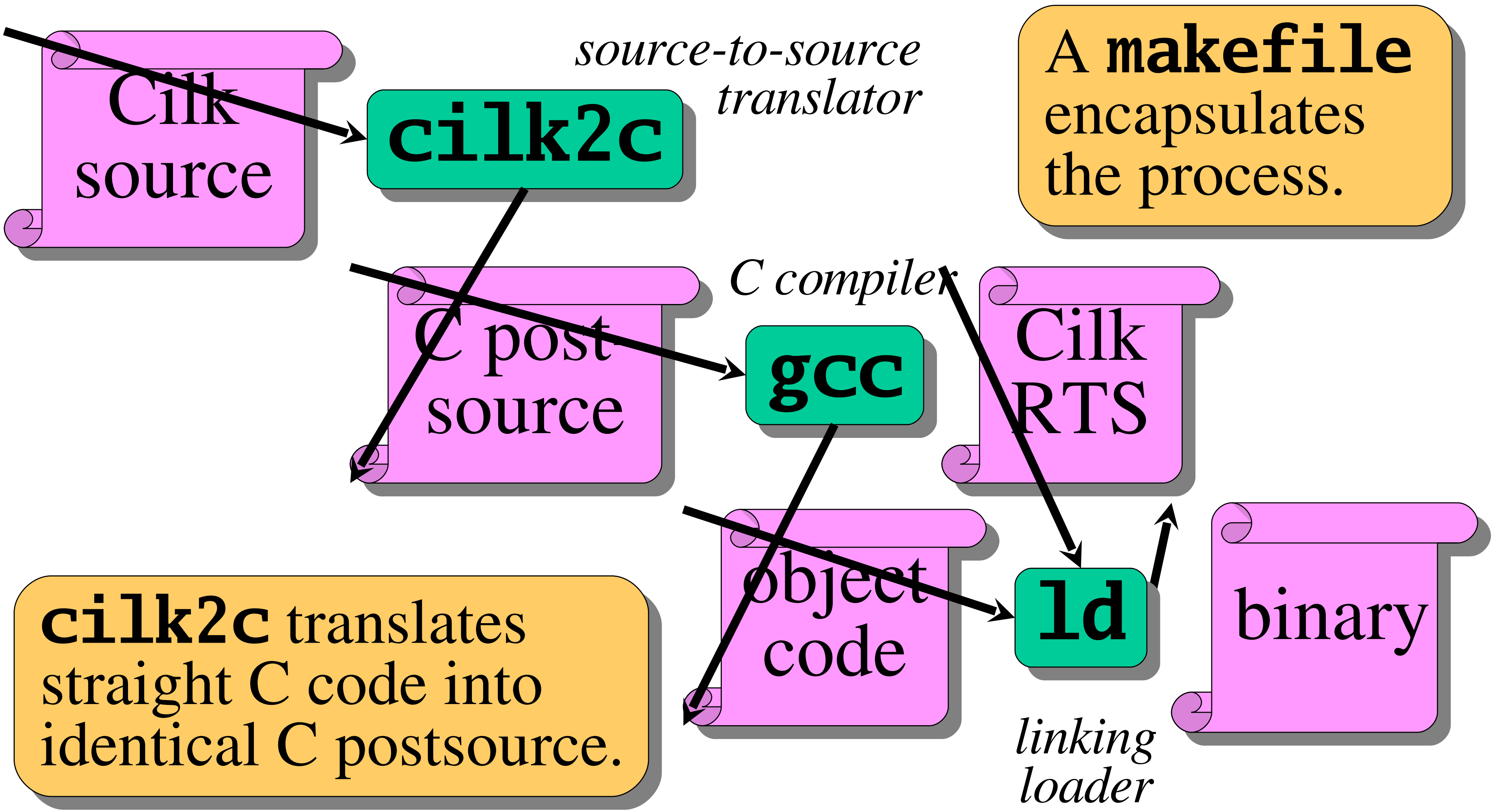
We have had success using functional languages to generate high-performance portable C codes.

- **FFTW**: *The Fastest Fourier Transform in the West* [Frigo-Johnson 1997]: 2–5£ vendor libraries.
- Divide-and-conquer strategy optimizes cache use.
- A special-purpose compiler written in Objective CAML optimizes FFT dag for each recursive level.
- At runtime, FFTW measures the performance of various execution strategies and then uses dynamic programming to determine a good execution plan.

http://theory.lcs.mit.edu/~fftw


Sacred Cow

Compiling Cilk

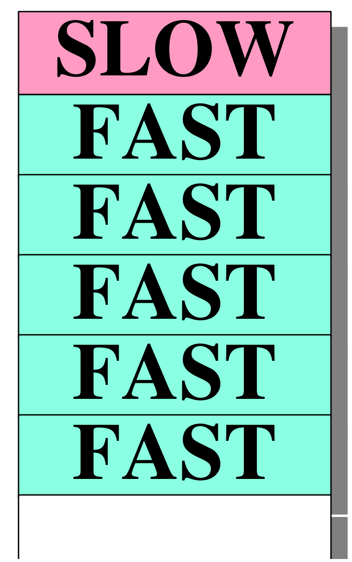


Cilk's Compiler Strategy

The **`cilk2c`** compiler generates two "clones" of each procedure:

- ***fast clone***—serial, common-case code.
- ***slow clone***—code with parallel bookkeeping.

- The *fast clone* is always spawned, saving live variables on Cilk's work deque (shadow stack).
- The *slow clone* is resumed if a thread is stolen, restoring variables from the shadow stack.
- A check is made whenever a procedure returns to see if the resuming parent has been stolen.



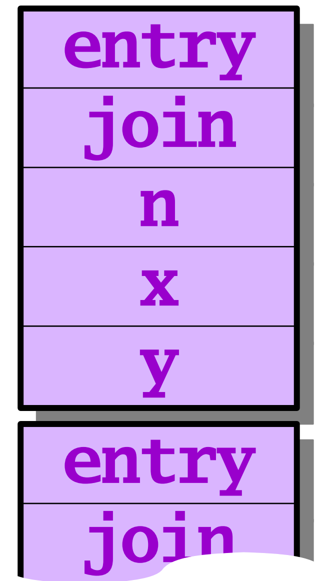
Compiling spawn (Fast Clone)

Cilk source

```
x = spawn fib(n-1);
```

cilk2c

frame



C post-source

```
frame->entry = 1;
frame->n = n;
push(frame);
```

suspend parent

```
x = fib(n-1);
```

run child

```
if (pop() == FAILURE)
   {   frame->x = x;
       frame->join--;
       h clean up & return
         to scheduler i     }
```

resume parent remotely

Compiling **sync** (Fast Clone)

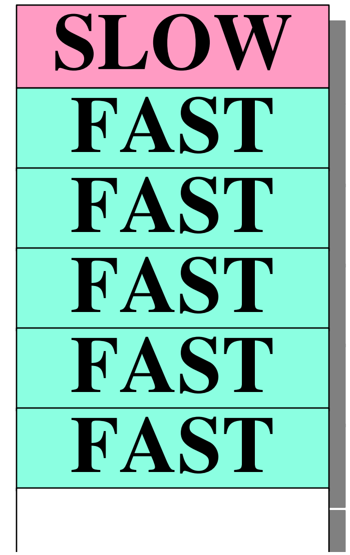
Cilk source

```
sync;
```

cilk2c

C post-source

```
;
```



No synchronization overhead in the fast clone!

Compiling the Slow Clone

```
void fib_slow(fib_frame *frame)
{
  int n,x,y;
  switch (frame->entry) {
    case 1: goto L1;
    case 2: goto L2;
    case 3: goto L3;
    }
        . . .
  frame->entry = 1;
  frame->n = n;
  push(frame);
  x = fib(n-1);
  if (pop() == FAILURE)
    {  frame->x = x;
       frame->join--;
       h clean up & return
         to scheduler i     }

  if (0) {
     L1:;
     n = frame->n;
  }
        . . .
}
```

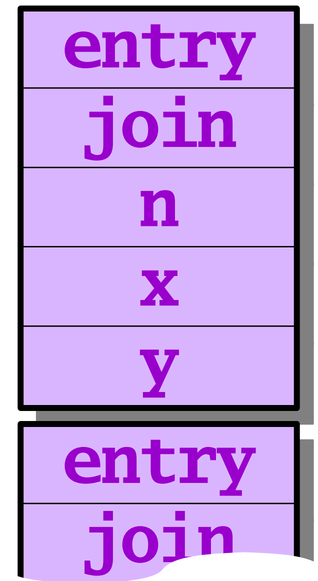
restore program counter

same as fast clone

restore local variables if resuming

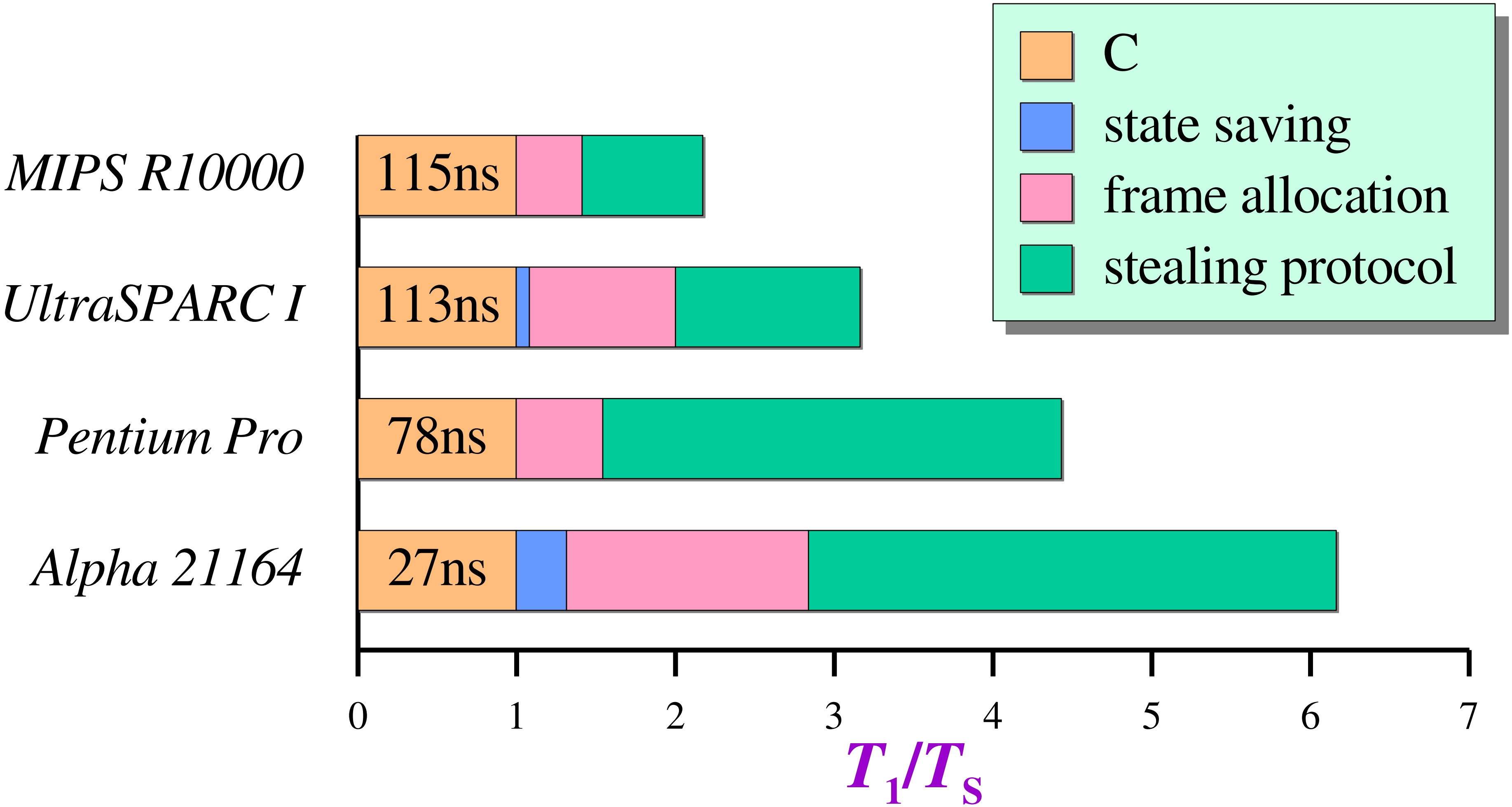
continue

frame



Cilk deque

Breakdown of Work Overhead



Benchmark: **`fib`** on one processor.

Mergesorting

```
cilk void Mergesort(int A[], int p, int r)
{
    int q;
    if ( p < r )
        {
            q = (p+r)/2;
            spawn Mergesort(A,p,q);
            spawn Mergesort(A,q+1,r);
            sync;
            Merge(A,p,q,r);    // linear time
        }
}
```

$T_1(n) = 2\,T_1(n/2) + (n)$

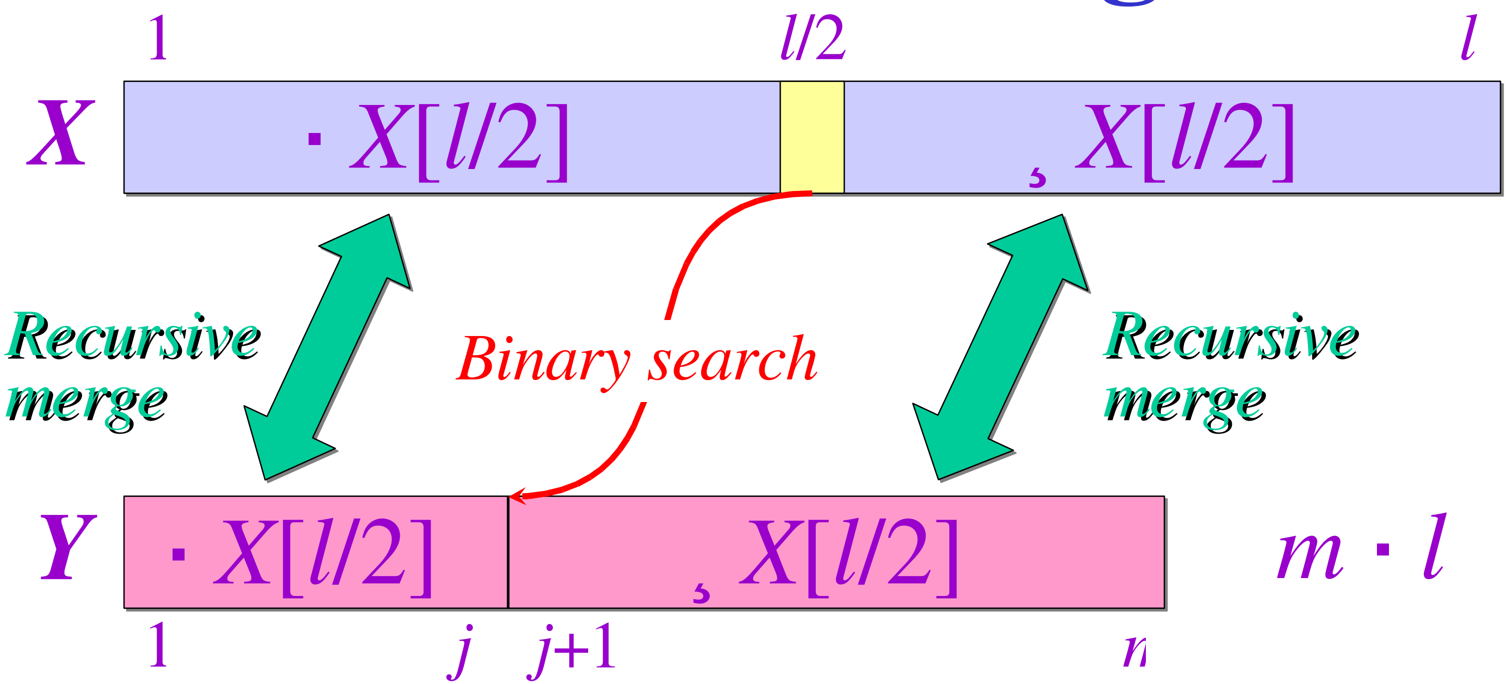
$T_\infty(n) = T_\infty(n/2) + (n)$ $(n \lg n)$

$= (n)$

Parallelism:

$$\frac{(n \lg n)}{(n)} = (\lg n)$$

Parallel Merge



$T_1(n) = T_1(\quad n) + T_1((1- \quad)n) + \quad (\lg n)$, where $1/4 \cdot \quad \cdot 3/4$

$= \quad (n)$

$T_\infty(n) = T_\infty(3n/4) + \quad (\lg n)$

$= \quad (\lg^2 n)$

Parallel Mergesort

$$T_1(n) = 2\,T_1(n/2) + \quad (n)$$
$$= \quad (n \lg n)$$

$$T_\infty(n) = T_\infty(n/2) + \quad (\lg^2 n)$$
$$= \quad (\lg^3 n)$$

Parallelism:

$$\frac{(n \lg n)}{(\lg^3 n)} = \quad (n/\lg^2 n)$$

- Our implementation of this algorithm yields a 21% work overhead and achieves a 6 times speedup on 8 processors (saturating the bus).
- Parallelism of $\quad (n/\lg n)$ can be obtained at the cost of increasing the work by a constant factor.

Student Assignment

Implement the fastest 1000 £ 1000 matrix-multiplication algorithm.

- ***Winner*:** A variant of ***Strassen's algorithm*** which permuted the row-major input matrix into a bit-interleaved order before the calculation.
- ***Losers*:** Half the groups had ***race bugs***, because they didn't bother to run the Nondeterminator.
- ***Learners:*** Should have taught ***high-performance C*** programming first. The students spent most of their time optimizing the serial C code and little of their time Cilkifying it.

Caching Behavior

Cilk's scheduler guarantees that

$$Q_P/P \cdot Q_1/P + O(MT_\infty/B) \ ,$$

where Q_P is the total number of cache faults on P processors, each with a cache of size M and cache-line length B.

Divide-and-conquer "cache-oblivious" matrix multiplication has

$$Q_1(n) = O(1 + n^3/\sqrt{MB}) \ ,$$

which is asymptotically optimal.

IDEA: *Once a submatrix fits in cache, no further cache misses on its submatrices.*

